LMT Fall 2024 Guts Round Solutions- Part 1

Team Name:

1. **[9]** Find the least prime factor of $2024^{2024}-1$.

Proposed by: Samuel Tsui

Solution. $\boxed{3}$

Note the number is odd and furthermore $2024^{2024}-1 \equiv (-1)^{2024}-1 \equiv 0 \pmod{\boxed{3}}$. □

2. **[9]** A group of nine math team members like to play Survev.io. They noticed that the number of hours each of them played this week forms an arithmetic progression. The person who played the least played for 1 hour, while the most played for 9. Find the total number of hours all nine group members spent playing Survev.io this week.

Proposed by: Lena Lee

Solution. $\boxed{45}$

The average number of hours played is $\frac{9+1}{2}=5$. Since there are 9 people the total number of hours is $9\cdot 5 = \boxed{45}$. □

3. **[9]** Two distinct positive even integers sum to 8. Find the larger of the two integers.

Proposed by: Edwin Zhao

Solution. $\boxed{6}$

The numbers must be 2 and 6, for an answer of $\boxed{6}$. □

LMT Fall 2024 Guts Round Solutions- Part 2

Team Name:

4. **[10]** A group of 5 rappers wants to make a song together. They each make their own parts for the song and then arrange the 5 parts. J Cole wants to be friends with both Drake and Kendrick, so he wants his part to be adjacent to both of theirs. Find the number of possible songs (distinct orders) that can be made.

Proposed by: Rohan Danda

Solution. $\boxed{12}$

J Cole has 3 possible locations. Then Drake and Kendrick have 2 ways to arrange themselves. The remaining two rappers have 2 ways, for an answer of $\boxed{12}$. □

5. **[10]** Ben Y's favorite number p is prime, and his second favorite number is some integer n. Given that p divides n and n divides $3p+91$, find the maximum possible value of n.

Proposed by: William Hua

Solution. $\boxed{130}$

We see that $p \mid n \mid 3p+91$ so $p \mid 3p+91$, meaning p must divide 91. Then $p=7,13$. Let $p=13$. then, $n \le 3\cdot 13+91 = \boxed{130}$. □

6. **[10]** Let P be a point in rectangle $ABCD$ such that the area of PAB is 20 and the area of PCD is 24. Find the area of $ABCD$.

Proposed by: Samuel Tsui

Solution. $\boxed{88}$

Let h_1 and h_2 be the heights from P to AB and CD, respectively. Note that

$$[PAB]+[PCD]=\frac{AB\cdot h_1}{2}+\frac{CD\cdot h_2}{2}=\frac{AB}{2}(h_1+h_2)=\frac{AB\cdot BC}{2}.$$

So $20+24$ is half the area, meaning the area is $2(20+24)=\boxed{88}$. □

LMT Fall 2024 Guts Round Solutions- Part 3

Team Name: ______________________

______ 7. **[11]** Let L be the answer to problem 9. Find the solution to the equation $4x+\sqrt{L}=0$.

Proposed by: Samuel Tsui

Solution. $\boxed{-1}$

See solution below □

______ 8. **[11]** Let M be the answer to problem 7. Let $f(x)=x^4+4x^3+6x^2+1$. Find $f(M)$.

Proposed by: Samuel Tsui

Solution. $\boxed{4}$

See solution below □

______ 9. **[11]** Let T be the answer to problem 8. Find the area of a square with side length T.

Proposed by: Samuel Tsui

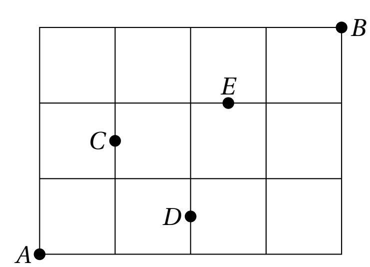
Solution. $\boxed{16}$

$M=-\frac{1}{4}\sqrt{L}, T=M^4+4M^3+6M^2+1, L=T^2$ and solving gives $(M+1)^4=0$ so $M=-1,\ T=4,$ $L=16$. □

LMT Fall 2024 Guts Round Solutions- Part 4

Team Name: ______________________

______ 10. **[12]** David starts at the point A and goes up and right along the grid lines to point B. At each of the points C, D, and E there is a bully. Find the number of paths David can take which make him encounter exactly one bully.



Proposed by: David Kim

Solution. $\boxed{14}$

We can split into cases depending on which point we visit.

- David goes to C. There are two ways to get to C and 2 ways to get from C to B. This gives 4 paths.
- David goes to D. There is one way to get to D. From there David has 4 ways to get to B.
- David goes E. There are 3 ways to get to E. From there David has 2 ways to get to B. This gives 6 paths.

In total we have $4+4+6=\boxed{14}$ paths. □

______ 11. **[12]** A Pokemon fan walks into a store. An employee tells them that there are 2 Pikachus, 3 Eevees, 4 Snorlaxes, and 5 Bulbasaurs remaining inside the gacha machine. Given that this fan cannot see what is inside the Poké Balls before opening them, find the least number of Poké Balls they must buy in order to be sure to get one Pikachu and one Snorlax.

Proposed by: Lena Lee

Solution. $\boxed{13}$

In the worst case, this fan will get 3 Eevees, then 5 Bulbasaurs, then 4 Snorlaxes, then finally their first Pikachu. This totals $3+4+5+1=\boxed{13}$ Poke Balls before their first guaranteed Pikachu and Snorlax. □

______ 12. **[12]** Snorlax's weight is modeled by the function $w(t) = t2^t$ where $w(t)$ is Snorlax's weight at time t minutes. Find the smallest integer time t such that Snorlax's weight is greater than 10000.

Proposed by: Benjamin Yin

Solution. $\boxed{10}$

Solving the inequality $t2^t > 10000$, the first integer t that satisfies it is $\boxed{10}$. □

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LMT Fall 2024 Guts Round Solutions- Part 5

Team Name:

13. **[13]** Suppose j, x, and u are positive real numbers such that $jxu = 20$ and $x + u = 24$. Find the minimum possible value of $j \max(x, u)$.

Proposed by: Evin Liang

Solution. $\boxed{\frac{5}{3}}$

Note that since $jxu = 20$, $j \max(x, u) = \frac{24}{\min(x,u)}$. Since $\min(x, u) \leq 12$, $j \max(x, u) \geq \frac{20}{12} = \frac{5}{3}$. Equality occurs when $j = \frac{5}{36}$ and $x = u = 12$. □

14. **[13]** Find the number of trailing 0s in the base 12 expression of 99! (Note: 99 is in base 10).

Proposed by: Sylvia Lee, Kalina Liu, Mira Xu

Solution. $\boxed{47}$

We have $v_2(99!) = 49 + 24 + 12 + 6 + 3 + 1 = 95$ and $v_3(99!) = 33 + 11 + 3 + 1 = 48$. So the largest power of 12 that divides 99! is $\min\left(\frac{95}{2}, 48\right) = \boxed{47}$. □

15. **[13]** Regular hexagon $ABCDEF$ with side length 2 is inscribed within a sphere of radius 4. Let point X be on the sphere. Find the maximum value of the volume of the pyramid $ABCDEFX$.

Proposed by: Calvin Garces

Solution. $\boxed{12 + 8\sqrt{3}}$

The area of the hexagon is $\frac{3(2)^2\sqrt{3}}{2} = 6\sqrt{3}$. The maximum height is $4 + \sqrt{4^2 - 2^2} = 4 + 2\sqrt{3}$. Thus the maximum volume is $\boxed{12 + 8\sqrt{3}}$. □

LMT Fall 2024 Guts Round Solutions- Part 6

Team Name:

16. **[14]** A new meme is circling around social media known as the *DaDerek Convertible*. The license plate number of the *DaDerek Convertible* is such that the product of its nonzero digits times 5 is equal to itself. Given that its license plate number has less than or equal to 3 digits and that it has at least one nonzero digit, find the *DaDerek Convertible*'s license plate number.

Proposed by: Edwin Zhao and Chris Cheng

Solution. $\boxed{175}$

The license plate number must be divisible by 5 so it ends in a 0 or a 5.

Case 1: the last digit is 5. The product of the digits must then be divisible by 5, so the number is divisible by 25. This means it ends with 25 or 75. It cannot end with 25 otherwise the last digit would be 0. If if ends with 75, the number is divisible by 175. The only such numbers are 175 and 875. Clearly, 175 works and 875 doesn't.

Case 2: the last digit is 0. Consider the 3-digit number $\overline{ab0}$. By the problem statement $a \cdot b \cdot 5 = \overline{ab0}$. This means $a \cdot b \cdot 5$ is at least $\overline{a00}$ which means $b \cdot 5 \geq 100$. However, b is a digit so this is clearly impossible. If the number is 2 digit, it is obvious $10, 20, \ldots, 90$ don't work. The number cannot be one digit because the problem requires at least one nonzero digit.

Thus, 175 is the only number that works. □

17. **[14]** Suppose x, y, z are pairwise distinct real numbers satisfying

$$x^2+3y=y^2+3z=z^2+3x.$$

Find $(x+y)(y+z)(z+x)$.

Proposed by: Muztaba Syed

Solution. $\boxed{-27}$

Subtracting the equations in pairs we get

$$\begin{aligned}(x-y)(x+y)&=3(z-y)\\(y-z)(y+z)&=3(x-z)\\(z-x)(z+x)&=3(y-x).\end{aligned}$$

Multiplying these equations, canceling $(x-y)(y-z)(z-x)$, and adjusting the sign, we get $-3^3=\boxed{-27}$. □

18. **[14]** In the electoral college, each of 51 places get some positive number of electoral votes for a nationwide total of 538. Thus, 270 electoral votes guarantees a win. Across all distributions of electoral votes to each place, let M be the maximum number of sets of places that combine to have at least 270 electoral votes. Find M.

Proposed by: Chris Cheng

Solution. $\boxed{2^{50}}$

Split the places into 2 subsets, either both subsets have 269 votes, or one subset has at least 270 and the other has less than 270. Thus, the goal is to minimize the number of subsets that would result in 269 votes each. This would give an answer of $\boxed{2^{50}}$ because half of the resulting subsets would have at least 270 votes, and there are 2^{51} total subsets. Minimizing is easy because it is possible to make it 0 by just giving one singular place more than 269 electoral votes. □

LMT Fall 2024 Guts Round Solutions- Part 7

Team Name:

19. **[15]** Given $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, find

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{1}{ij(i+1)(j+1)}.$$

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{\pi^2}{6}-1}$

Swapping order of summation gives

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{1}{ij(i+1)(j+1)}=\sum_{i=1}^{\infty}\frac{1}{i(i+1)}\sum_{j=i}^{\infty}\frac{1}{j(j+1)}=\sum_{i=1}^{\infty}\frac{1}{i^2(i+1)}=\sum_{i=1}^{\infty}\left(-\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n^2}\right)=\boxed{\frac{\pi^2}{6}-1}.$$

□

20. **[15]** A base 9 number *probably places* if it has a 7 as one of its digits. Find the number of base 9 numbers less than or equal to 100 in base 10 that probably place.

Proposed by: Jerry Xu

Solution. $\boxed{19}$

The numbers from 1 to 100 in base 10 correspond from 1 to 121 in base 9. We now split into cases based on the number of digits in the base 9 representation:

- 1 digit: there is exactly one number which has 7 as a digit, namely 7 itself.
- 2 digits: there are 8 numbers that end in 7 from 17 to 87, and 9 numbers that start with 7 from 70 to 78. However, this overcounts 77 once, for a total of $8+9-1=16$ numbers which have a 7 as a digit.
- 3 digits: the only numbers which have a 7 as a digit are 107 and 117.

Hence, our total is $1+16+2=\boxed{19}$. □

______ 21. **[15]** Let ABC be a triangle such that $AB=2$, $BC=3$, and $AC=4$. A circle passing through A intersects AB at D, AC at E, and BC at M and N such that $BM=MN=NC$. Find DE.

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{3\sqrt{15}}{4}}$

Power of a point gives $BD\cdot 2=1\cdot 2$ and $CE\cdot 4=1\cdot 2$ thus $BD=1$, $CE=\frac{1}{2}$ so $AD=1$, $AE=\frac{7}{2}$. From law of cosines on ABC, $\cos\angle A=\frac{2^2+4^2-3^2}{2\cdot 2\cdot 4}=\frac{11}{16}$. Law of cosines again on ADE gives $DE^2=1^2+\frac{7}{2}^2-2\cdot 1\cdot\frac{7}{2}\cdot\frac{11}{16}=\frac{135}{16}$ thus $DE=\boxed{\frac{3\sqrt{15}}{4}}$. □

LMT Fall 2024 Guts Round Solutions- Part 8

Team Name:

______ 22. **[17]** Find the number of real numbers $0\le\alpha<50$ such that $\alpha^2+2\{\alpha\}$ is an integer. (Here $\{\alpha\}$ denotes the fractional part of α.)

Proposed by: Ryan Tang

Solution. $\boxed{2600}$

The problem is equivalent to finding the number of real numbers such that $\alpha^2+2\alpha$ is an integer, since $2\lfloor\alpha\rfloor\in\mathbb{Z}$, and we can add this to our expression. Note that $\alpha^2+2\alpha$ is bounded by 0 and 2600 by our given. It is easy to see that each value is attainable by exactly one value of α in our range. However, note that the problem is asking for a positive integer, and so 0 clearly doesn't work. Thus, the answer is $\boxed{2600}$. □

______ 23. **[17]** Define $\overline{a}$ of a positive integer a to be the number a with its digits reversed. For example, $\overline{31564}=46513$. Find the sum of all positive integers $n\le 100$ such that $(\overline{n})^2=\overline{n^2}$. (Note: For a number that ends with a zero, like 450, the reverse would exclude the zero, so $\overline{450}=54$).

Proposed by: Benjamin Yin

Solution. $\boxed{176}$

First checking single digit numbers, we find that $n=1,2,3$ work. Next, notice that adding a zero to the end of a number m does not change $\overline{m}$ or $\overline{m^2}$, thus, the values $n=10,20,30$ also work. Finally, we consider numbers of the form $n=10a+b$, where a and b are positive digits. We split into two cases.

Case 1: Three Digit Squares. We first consider numbers that have a square with three digits. Notice that $31^2<1000<32^2$, so all numbers that have three digits in their square must be less than 32. So, every n in this case must have both digits less than 4 (if one digit is greater than or equal to 4, having that digit in the tens spot will result in a square with greater than three digits). The numbers that satisfy these conditions are $11, 12, 13, 21, 22, 23, 31, 32, 33$. Out of these, it is easy to check that only $11, 12, 13, 21, 22$, and 31 work.

Case 2: Four Digit Squares. For this case, we further split into cases based on first digit of the square. Notice that in order for a certain n to work, the leading digit of its square must be able to be the units digit of a square (when n^2 is reversed the leading digit becomes the unit digit). The digits $1, 4, 5, 6, 9$ are the only possible units digits for a square number (we are excluding 0, as we have already considered that earlier on).

n	First Digit of n^2
32 to 44	1
64 to 70	4
71 to 77	5
78 to 83	6
95 to 99	9

Now, notice that the first digit of n^2 must be the units digit of $(\overline{n})^2$, however, going through all the values of n in our table, we notice that none of them satisfy our property (for example, from 32 to 44, when n is reversed, the units digit is either 3 or 4, and when squared, gives 9 or 6, which is not equal to 1). Thus, there are no solutions for this case. Our final answer is $1+2+3+10+20+30+11+12+13+21+22+31 = \boxed{176}$. $\square$

24. **[17]** Let ABC be a triangle with $AB = 13, BC = 15, AC = 14$. Let P be the point such that $AP = CP = \frac{1}{2}BP$. Find AP^2.

Proposed by: Benjamin Yin

Solution. $\boxed{65}$

Since $AP = CP$, point P must lie on the perpendicular bisector of AC. Let F be the foot of the perpendicular from B to AC and let M be the midpoint of AC. It is well known that $BF = 12, AF = 5$, so we also have $FM = AM - AF = 2$. Now, by Pythagorean Theorem, we compute $BM = \sqrt{12^2 + 2^2} = \sqrt{148} < 14$. Since $\frac{1}{2}BP < 7$ we know that point P must lie on the opposite side of AC from point B. Let $MP = x$, from Pythagorean Theorem, we have $BP = \sqrt{(x+12)^2 + 2^2}$ $AP = \sqrt{x^2 + 7^2}$ Since $BP = 2AP$, we have $4(x^2+49) = (x+12)^2+4$ $4x^2+196 = x^2+24x+148$ $3x^2-24x+48 = 0$ $x^2-8x+16 = 0$ $(x-4)^2 = 0$ $x = 4$ Thus, we have $AP^2 = 7^2 + 4^2 = \boxed{65}$.

Alternatively since $2BC = AB + AC$ it is well known (by Ptolemy's Theorem) that the midpoint M of the minor arc BC satisfies $MA = 2MB = 2MC$. $\square$

_______ 25. **[20]** Define $f(n)$ to be the sum of positive integers k less than or equal to n such that $\gcd(n,k)$ is prime. Find $f(2024)$.

Proposed by: Henry Eriksson

Solution. $\boxed{574816}$

For $p \mid n$, there are $\phi\left(\frac{n}{p}\right)$ numbers less than or equal to n such that $\gcd(n,k) = p$, because of the bijection $k \to \frac{k}{p}$ between these numbers and numbers counted in $\phi\left(\frac{n}{p}\right)$. The sum of $k \le n$ such that $\gcd(n,k) = p$ is p times the sum of numbers less than or equal to $\frac{n}{p}$ and coprime to $\frac{n}{p}$. For all i coprime to $\frac{n}{p}$ and $\frac{n}{p} > 1$, we can pair i and $\frac{n}{p} - i$, which are different and both coprime to $\frac{n}{p}$, so the sum of numbers less than or equal to $\frac{n}{p}$ and coprime to $\frac{n}{p}$ is $\frac{\frac{n}{p}\phi\left(\frac{n}{p}\right)}{2}$, so the sum of $k \le n$ such that $\gcd(n,k) = p$ is $p \cdot \frac{\frac{n}{p}\phi\left(\frac{n}{p}\right)}{2} = \frac{n\phi\left(\frac{n}{p}\right)}{2}$, so

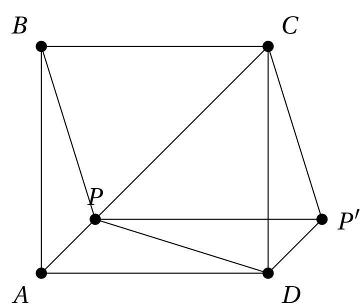
$$f(n) = \sum_{p|n} \frac{n\phi\left(\frac{n}{p}\right)}{2} = \frac{n}{2}\sum_{p|n} \phi\left(\frac{n}{p}\right).$$

For $n = 2024 = 2^3 \cdot 11 \cdot 23$, we find this to be $1012 \cdot (440+88+40) = 1012 \cdot 568 = 574816$. □

_______ 26. **[20]** Let P be a point in the interior of square $ABCD$ such that $\angle APB + \angle CPD = 180^\circ$ and $\angle APB < \angle CPD$. If $PC = 7$ and $PD = 5$, find $\frac{PA}{PB}$.

Proposed by: Jerry Xu

Solution. $\boxed{\dfrac{7\sqrt{73}}{73}}$



Claim: P lies on a diagonal of $ABCD$.

Proof: Fix α. We claim there exists exactly two points P_1, P_2 for which $\angle AP_iB = \alpha$ and $\angle CP_iD = 180^\circ - \alpha$. But this is clear, since P_i must lie on the circle through A and B with minor arc AB subtending α and the circle through C and D with minor arc CD subtending $180^\circ - \alpha$, and two circles intersect at most twice. Now, let Y be the point on AC such that $\angle AYC = \angle APB$. It is easy to see that $BY = DY$, so $ABY \cong ADY$ and $CBY \cong CDY$, from which we determine

$$\angle CYD = 180^\circ - \angle DYA = 180^\circ - \angle AYB,$$

and similar logic shows that some Z on BD also satisfies the supplementary condition. Hence, we have proven that both possibilities for P lie on a diagonal. ■

WLOG, assume P lies on BD, so that $PA = PC = 7$. We finish with a translation: Let the side length of $ABCD$ be s. Translate APB by s so that AB coincides with CD. Let P' denote the image of P under this translation, so that $P'C = PB$ and $P'D = PA$. Then since

$$\angle CPD + \angle CP'D = \angle CPD + \angle APB = 180^\circ,$$

$PCP'D$ is cyclic. Hence, by Ptolemy's,

$$s^2 = PP' \cdot CD = PC \cdot P'D + PD \cdot P'C = 49 + 5PB.$$

Finally, we can note that $PB = s\sqrt{2} - 5$. Solving the quadratic gives $PB = \sqrt{73}$, so our answer is $\boxed{\frac{7\sqrt{73}}{73}}$. □

__________ 27. **[20]** Find all positive integer pairs (a, b) that satisfy the equation

$$a^2b + ab^2 + 73 = 8ab + 9a + 9b.$$

Proposed by: Samuel Tsui

Solution. $\boxed{(1,8),(8,1),(2,5),(5,2)}$

Let $x = a + b$ and $y = ab$. Then $(x-8)(y-9) = -1$ so $(x, y) = (7,10), (9,8)$. This gives the ordered pairs of $\boxed{(1,8),(8,1),(2,5),(5,2)}$ □

LMT Fall 2024 Guts Round Solutions- Part 10

Team Name: ____________________

__________ 28. **[23]** Find the number of ways to tile a $2 \times 2 \times 2 \times 2$ four dimensional hypercube with $2 \times 1 \times 1 \times 1$ blocks, with reflections and rotations of the large hypercube distinct.

Proposed by: Edwin Zhao

Solution. $\boxed{272}$

We casework on how many blocks stay inside their own 3 dimensional cube.

- 0: Each block must connect to their mirror, so this is just 1.
- 1: 12 ways to select the block that is in their own cube, everything else connects to their mirror.
- 2: If they make up a 2×2 square, there are 6 ways to choose the square then 2 ways to arrange the blocks inside that square so $6 \cdot 2^2 = 24$. If they don't make up a 2×2 square there are 12 ways to choose the first block and 5 ways to choose the 2nd block such that it doesn't form a 2×2 square. This gives $12 \cdot 5 \div 2 = 30$ ways. Thus this case has $24 + 30 = 54$ possibilities.
- 3: Assume the 2 mirrored squares are next to each other. 12 ways to select this, then 3 ways to orient the last 3 blocks so that they fit inside the shape so $12 \cdot 3^2 = 108$. If they aren't next to each other then the mirrored blocks must be in opposite corners of the cube. There are 4 ways to select these opposite corners, and 2 ways to orient last 3 blocks so they form the weird snaking path. This gives $4 \cdot 2^2 = 16$, for a total of $16 + 108 = 124$.
- 4: We see that this is 9 for each cube for a total of $9^2 = 81$.

Our answer is then $1 + 12 + 54 + 124 + 81 = \boxed{272}$ □

__________ 29. **[23]** Let $P(x)$ be a quartic polynomial with integer coefficients and leading coefficient 1 such that $P(\sqrt{2} + \sqrt{3} + \sqrt{6}) = 0$. Find $P(1)$.

Proposed by: Evin Liang

Solution. $\boxed{-92}$

In fact the polynomial is

$$P(x) = (x - \sqrt{2} - \sqrt{3} - \sqrt{6})(x - \sqrt{2} + \sqrt{3} + \sqrt{6})(x + \sqrt{2} - \sqrt{3} + \sqrt{6})(x + \sqrt{2} + \sqrt{3} - \sqrt{6}).$$

Let $x = \sqrt{2} + \sqrt{3} + \sqrt{6}$. Then note that

$$x^2 = 11 + 2\sqrt{6} + 6\sqrt{2} + 4\sqrt{3}.$$

So we see

$$\frac{x^2 - 11}{2} = \sqrt{6} + 3\sqrt{2} + 2\sqrt{3}.$$

Squaring this, we get

$$\left(\frac{x^2 - 11}{2}\right)^2 = 6 + 18 + 12 + 2(6\sqrt{2} + 6\sqrt{3} + 6\sqrt{6}) = 36 + 12x.$$

Thus we have

$$\left(\frac{x^2 - 11}{2}\right)^2 = 36 + 12x,$$

from which we obtain

$$x^4 - 22x^2 - 48x - 23 = 0.$$

Note that the minimal polynomial of x should have degree 4, so $P(x) = x^4 - 22x^2 - 48x - 23$. Plugging in 1 we get $\boxed{-92}$. □

30. **[23]** Find

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{(-4)^n - 1},$$

where $\varphi(n)$ is the number of positive integers $k \le n$ relatively prime to n. (Note $\varphi(1) = 1$.)

Proposed by: Jerry Xu

Solution. $\boxed{-\frac{4}{25}}$

We shall generalize this problem by replacing -4 with a for $|a| > 1$.

Claim:

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{a^n - 1} = \sum_{n=1}^{\infty} n a^{-n}.$$

Proof: Observe that

$$\begin{aligned}
\frac{k^n}{1 - k^n} &= k^n \frac{1}{1 - k^n} \\
&= k^n \left(1 + k^n + k^{2n} + \cdots\right) \\
&= k^n + k^{2n} + k^{3n} + \ldots
\end{aligned}$$

For all $|k| < 1$. Now, we perform black magic. Since, $\left|\frac{1}{a}\right| < 1$, we have

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{a^n - 1} = \sum_{n=1}^{\infty} \varphi(n) \frac{a^{-n}}{1 - a^{-n}} = \sum_{n=1}^{\infty} \varphi(n) \left(a^{-n} + a^{-2n} + \cdots\right).$$

Instead of grouping terms by $\varphi(n)$, group terms by powers of a. We obtain

$$\sum_{n=1}^{\infty} \varphi(n) \left(a^{-n} + a^{-2n} + \cdots\right) = \sum_{n=1}^{\infty} a^{-n} \sum_{d|n} \varphi(d).$$

Now, recalling that $\sum_{d|n} \varphi(d) = n$, we obtain the desired.

It remains to evaluate the resulting arithmetico-geometric series:

$$\sum_{n=1}^{\infty} na^{-n} = \frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3} + \cdots$$
$$\frac{1}{a}\sum_{n=1}^{\infty} na^{-n} = \frac{1}{a^2} + \frac{2}{a^3} + \frac{3}{a^4} + \cdots$$
$$\left(1 - \frac{1}{a}\right)\sum_{n=1}^{\infty} na^{-n} = \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \cdots$$
$$= \frac{\frac{1}{a}}{1 - \frac{1}{a}} = \frac{1}{a-1}.$$

We thus obtain

$$\sum_{n=1}^{\infty} na^{-n} = \frac{1}{a-1} \cdot \frac{a}{a-1} = \frac{a}{(a-1)^2}.$$

Substituting in $a = -4$, we get an answer of $\boxed{-\frac{4}{25}}$. □

LMT Fall 2024 Guts Round Solutions- Part 11

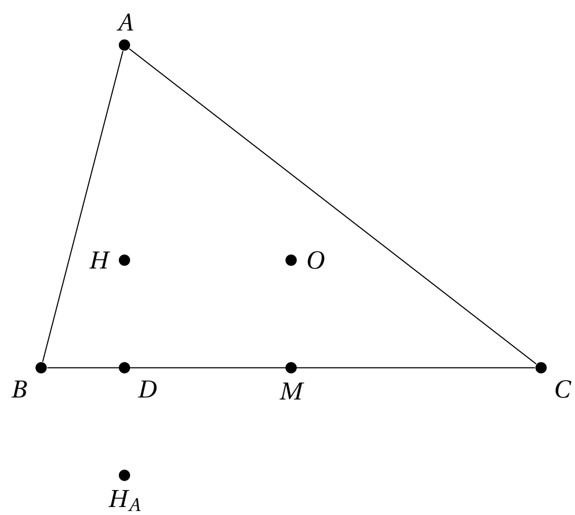
Team Name: ______________________________

______________ 31. **[26]** Let ABC be a triangle with circumradius 12, and denote the orthocenter and circumcenter as H and O respectively. Define $H_A \neq A$ to be the intersection of line AH and the circumcircle of ABC. Given that $\overline{OH} \parallel \overline{BC}$ and $\overline{AO} \parallel \overline{BH_A}$, find AH_A.

Proposed by: Adam Ge

Solution. $\boxed{6\sqrt{10}}$

Let D be the foot of the A-altitude, G be the centroid of ABC, and M be the midpoint of BC. Let $HD = a$ and $BD = b$.



Since G lies on $\overline{OH}$ and $\overline{OH} \parallel \overline{BC}$, $\frac{AH}{HD} = 2$ so $DH_A = a$ and $AH = 2a$.

Now note that $\angle OAH = \angle DH_AB$ by the parallel condition, and $\angle AHO = \angle BDH_A = 90°$, so by AA similarity we have $AHO \sim H_ADB$. Thus $OH = 2b$.

Note that $HOMD$ is a rectangle, so $DM = OH = 2b$. By the Pythagorean Theorem on AHO, and OMB we see $4a^2 + 4b^2 = AO^2 = BO^2 = OM^2 + MB^2 = a^2 + 9b^2 \implies \frac{a}{b} = \frac{\sqrt{5}}{\sqrt{3}}$. We can then solve for a to get $AH_A = 4a = \frac{4a}{\sqrt{4a^2+4b^2}} \cdot AO = \frac{24a}{\sqrt{a^2+b^2}} = \boxed{6\sqrt{10}}$. □

______________ 32. **[26]** Let a and b be positive integers such that

$$a^2 + (a+1)^2 = b^4.$$

Find the least possible value of $a+b$.

Proposed by: Samuel Tsui and Ryan Tang

Solution. $\boxed{132}$

We will try to generate a Pythagorean triple that satisfies the equation. Without loss of generality suppose $m \geq n$ such that $m^2 - n^2 - 2mn = \pm 1$, or $(m-n)^2 - 2n^2 = \pm 1$. Considering the negative first, by Pell's Equation we have that $(m-n, n) = (1,1), (3,2), (7,5)$ work. Moreover, notice that $(m,n) = (12,5)$ yields $12^2 + 5^2 = 13^2$ giving the solution $a = 119, b = 13$. It can be checked that the positive case has no smaller solutions, thus the answer is $\boxed{132}$. □

______________ 33. **[26]** Let a and b be positive real numbers that satisfy

$$\sqrt{a-ab} + \sqrt{b-ab} = \frac{\sqrt{6}+\sqrt{2}}{4} \text{ and } \sqrt{a-a^2} + \sqrt{b-b^2} = \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2.$$

Find the ordered pair (a,b) such that $a > b$ and $a+b$ is maximal.

Proposed by: Selena Ge

Solution. $\boxed{\left(\frac{3}{4}, \frac{1}{2}\right)}$

First notice that $0 < a < 1$ and $0 < b < 1$, so we can replace a with $\sin^2\alpha$ and $\sin^2\beta$, where we assume without loss of generality that $0 < \alpha < 90$ and $0 < \beta < 90$. We get $\sin(\alpha+\beta) = \frac{\sqrt{6}+\sqrt{2}}{4}$ and $\sin 2\alpha + \sin 2\beta = 1 + \frac{\sqrt{3}}{2}$. By the sine addition formula, $\sin 2\alpha + \sin 2\beta = 2\sin(\alpha+\beta)\cos(\alpha-\beta) = \frac{\sqrt{6}+\sqrt{2}}{2}\cos(\alpha-\beta)$, so $\cos(\alpha-\beta) = \frac{\sqrt{6}+\sqrt{2}}{4}$. Thus due to the bounds we either have $\alpha+\beta = 75$ or 105 and $\alpha - \beta = -15$ or 15. Now, we can try the 4 possibilities and see that $\boxed{\left(\frac{3}{4}, \frac{1}{2}\right)}$ maximizes $a+b$ with $a > b$. □

34. **[30]** Let a sequence a_n be defined by $a_0 = 0$ and $a_n = na_{n-1} + 2^n$. Estimate $\frac{a_{2024}}{2024!}$.

Submit a positive real number E in decimal form. If the correct answer is A, you will receive $\min\left(30, \frac{5}{|E-A|}\right)$ points.

Proposed by: Evin Liang

Solution. $\boxed{e^2 - 1 \pm 10^{-5000}}$

Let $b_n = \frac{a_n}{n!}$. Then $b_n = b_{n-1} + \frac{2^n}{n!}$, so $b_n = \sum_{i=1}^n \frac{2^i}{i!}$. Thus $b_{2024} \approx \sum_{i=1}^\infty \frac{2^i}{i!} = e^2 - 1$. □

35. **[30]** Estimate the number of ways to tile a 5 dimensional $2\times2\times2\times2\times2$ *hypercube* with $2\times1\times1\times1\times1$ blocks, with reflections and rotations of the large cube distinct.

Submit a positive real number E in decimal form. If the correct answer is A, your score will be

$$\max\left(0, \left\lfloor 30\left(1 - \left|\ln\frac{E}{A}\right|\right)\right\rfloor\right).$$

Proposed by: Edwin Zhao

Solution. $\boxed{589185}$

It has to be at least 272^2 from the 4 dimensional case. Note that going from 3D to 4D is about $9^2 \cdot 3.5$ we can try doing $272^2 \cdot 3.5$ giving 258944 which on its own nets 5 out of the possible 30 points.

To optimize further also notice that going from 2D to 3D is about $2^2 \cdot 2$. This might suggest that we are going to want to multiply 272^2 by more than 3.5, perhaps times 8 which gives 29 out of the possible 30 points. □

36. **[30]** Stan, Kyle, Cartman, and Kenny stand at the four corners of South Park, a square with side length 1 mile. Each of them walks at a random angle between their two adjacent edges at 1 mile per hour. After 1 hour, Kenny spontaneously explodes and dies, killing everyone strictly within a half-mile of him. Estimate the expected number of people who will be killed in this explosion.

Submit a positive real number E in decimal form. If the correct answer is A, you will receive $\min\left(30, \left\lfloor \frac{1}{|E-A|^{0.6}} \right\rfloor\right)$ points.

Proposed by: Atticus Oliver

Solution. $\boxed{1.9345821}$

Strategy: First note that Kenny is a person and thus always contributes 1 to the expected value. Notice that Kenny can just end up killing someone adjacent to him by walking almost directly towards them, but doing so will almost certainly not kill anyone else. If he instead walks towards the middle, then it's feasible but not likely for everyone to avoid him, so the answer is a little less than 2. The exact number is given by

$$1 + \frac{4}{\pi^2}\left(\arccos\left(\frac{7}{8}\right)^2 + 8\int_{\arccos\left(\frac{1}{\sqrt{2}}+\frac{1}{4}\right)}^{\frac{\arcsin\left(\frac{7}{8}\right)}{2}} \arccos\left(\frac{7}{16\sqrt{2}\cos x} + \frac{\cos x}{\sqrt{2}}\right)dx\right.$$
$$+ 8\int_{\arccos\left(\frac{3}{4}\right)}^{\arccos\left(\frac{1}{4}\sqrt{\frac{5+\sqrt{7}}{2}}\right)} \arccos\left(\frac{3}{16\cos x} + \cos x\right)dx + 8\int_{\arccos\left(\frac{1}{4}\sqrt{\frac{5-\sqrt{7}}{2}}\right)}^{\arccos\left(\frac{1}{4}\right)} \arccos\left(\frac{3}{16\cos x} + \cos x\right)dx$$
$$\left. + 4\arcsin\left(\frac{1}{4}\sqrt{\frac{5+\sqrt{7}}{2}}\right)^2 - 4\arcsin\left(\frac{1}{4}\sqrt{\frac{5-\sqrt{7}}{2}}\right)^2\right).$$

For an answer of $\boxed{1.9345821}$. □